Two relations that generalize the q-Serre relations and the Dolan-Grady relations*

Paul Terwilliger

Abstract

We define an algebra on two generators which we call the *Tridiagonal algebra*, and we consider its irreducible modules. The algebra is defined as follows. Let $\mathbb{K}$ denote a field, and let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote a sequence of scalars taken from $\mathbb{K}$. The corresponding Tridiagonal algebra T is the associative $\mathbb{K}$-algebra with 1 generated by two symbols A, A^* subject to the relations

$$\begin{aligned}
[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] &= 0, \\
[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A] &= 0,
\end{aligned}$$

where $[r, s]$ means $rs - sr$. We call these relations the *Tridiagonal relations*. For $\beta = q + q^{-1}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 0$, the Tridiagonal relations are the q-Serre relations

$$\begin{aligned}
A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 &= 0, \\
A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} &= 0,
\end{aligned}$$

where $[3]_q = q + q^{-1} + 1$. For $\beta = 2$, $\gamma = \gamma^* = 0$, $\varrho = b^2$, $\varrho^* = b^{*2}$, the Tridiagonal relations are the Dolan-Grady relations

$$\begin{aligned}
[A, [A, [A, A^*]]] &= b^2[A, A^*], \\
[A^*, [A^*, [A^*, A]]] &= b^{*2}[A^*, A].
\end{aligned}$$

In the first part of this paper, we survey what is known about irreducible finite dimensional T-modules. We focus on how these modules are related to the Leonard pairs recently introduced by the present author, and the more general Tridiagonal pairs recently introduced by Ito, Tanabe, and the present author. In the second part of the paper, we construct an infinite dimensional irreducible T-module based on the Askey-Wilson polynomials. This module is on the vector space $\mathbb{K}[x]$ consisting of all polynomials in an indeterminant x that have coefficients in $\mathbb{K}$. Denoting by A the linear transformation on $\mathbb{K}[x]$ which is multiplication by x, and denoting by A^* an

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Askey-Wilson second order q-difference operator for x, we show A and A^* satisfy a pair of Tridiagonal relations. Using this we give $\mathbb{K}[x]$ the structure of an irreducible T-module. The Askey-Wilson polynomials form a basis for this module, and these basis elements are eigenvectors for A^*.

1 Leonard pairs

Throughout this paper, $\mathbb{K}$ will denote an arbitrary field.

We begin by recalling the notion of a Leonard pair.

Definition 1.1 *[54] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a Leonard pair on V, we mean an ordered pair A, A^*, where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy both (i), (ii) below.*

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

(A tridiagonal matrix is said to be irreducible whenever all entries immediately above and below the main diagonal are nonzero).

Note 1.2 *According to a common notational convention, for a linear transformation A the conjugate-transpose of A is denoted A^*. We emphasize we are not using this convention. In a Leonard pair A, A^*, the linear transformations A and A^* are arbitrary subject to (i), (ii) above.*

Here is an example of a Leonard pair. Set $V = \mathbb{K}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view A and A^* as linear transformations from V to V. We assume the characteristic of $\mathbb{K}$ is not 2 or 3, to ensure A is irreducible. Then A, A^* is a Leonard pair on V. Indeed, condition (ii) in Definition 1.1 is satisfied by the basis for V consisting of the columns of the 4 by 4 identity matrix. To verify condition (i), we display an invertible matrix P such that $P^{-1}AP$ is diagonal and $P^{-1}A^*P$ is irreducible tridiagonal. Set

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 8I$, where I denotes the identity, so P^{-1} exists. Also by matrix multiplication,

$$AP = PA^*. \tag{1}$$

Apparently $P^{-1}AP$ equals A^* and is therefore diagonal. By (1), and since P^{-1} is a scalar multiple of P, we find $P^{-1}A^*P$ equals A and is therefore irreducible tridiagonal. Now condition (i) of Definition 1.1 is satisfied by the basis for V consisting of the columns of P.

The above example is a member of the following infinite family of Leonard pairs. For any nonnegative integer d, the pair

$$A = \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \qquad A^* = \mathrm{diag}(d, d-2, d-4, \ldots, -d) \tag{2}$$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is zero or greater than d. This can be proved by modifying the proof for $d = 3$ given above. One shows $P^2 = 2^d I$ and $AP = PA^*$, where P denotes the matrix with ij entry

$$P_{ij} = \binom{d}{j} {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix}; 2\right) \qquad (0 \leq i, j \leq d). \tag{3}$$

We follow the standard notation for hypergeometric series given in [18].

In [54] we obtain a complete classification of Leonard pairs over an arbitrary field $\mathbb{K}$. We will not need the details here, but we wish to discuss one aspect. There is a connection between Leonard pairs and certain orthogonal polynomials contained in the Askey scheme [35]. Observe the ${}_2F_1$ that appears in (3) is a Krawtchouk polynomial [35]. There exist families of Leonard pairs similar to the one above in which the Krawtchouk polynomial is replaced by one of the following.

type	polynomial
${}_4F_3$	Racah
${}_3F_2$	Hahn, dual Hahn
${}_2F_1$	Krawtchouk
${}_4\phi_3$	q-Racah
${}_3\phi_2$	q-Hahn, dual q-Hahn
${}_2\phi_1$	q-Krawtchouk (classical, affine, quantum, dual)

The above polynomials are defined in Koekoek and Swarttouw [35], and the connection to Leonard pairs is given in [54, ch. 15] and [4, p. 260]. The above examples exhaust essentially all Leonard pairs in the following sense. As we will see below in the section on eigenvalues, associated with any Leonard pair is a certain scalar q. The above examples exhaust all Leonard pairs for which $q \neq -1$.

There is a theorem due to Leonard [41] and Bannai and Ito [4, p. 260] that gives a characterization of the polynomials from the above table for $\mathbb{K} = \mathbb{R}$. This result has come to be

known as Leonard's theorem. Our above mentioned classification of Leonard pairs amounts to a "linear algebraic version" of Leonard's theorem.

In this paper we are concerned with the following feature of Leonard pairs. For the Leonard pair in (2) one can show

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*, \tag{4}$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A. \tag{5}$$

This phenomenon is not a coincidence. As we will see, every Leonard pair satisfies two polynomial relations reminiscent of (4), (5). Before pursuing this further, we consider a certain generalization of a Leonard pair.

2 Tridiagonal pairs

In this section we shift our attention from a Leonard pair to a more general object called a *Tridiagonal pair*. To define this we use the following notation. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ denote a linear transformation. By an *eigenspace* of A we mean a nonzero subspace of V of the form

$$\{v \in V \mid Av = \theta v\},$$

where $\theta \in \mathbb{K}$. The scalar θ is the associated eigenvalue. We say A is *diagonalizable* on V whenever V is spanned by the eigenspaces of A.

Definition 2.1 *[33] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a Tridiagonal pair (or TD pair) on V we mean an ordered pair A, A^*, where $A : V \to V$ and $A^* : V \to V$ are linear transformations satisfying (i)–(iv) below.*

(i) A and A^* are both diagonalizable on V.

(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d), \tag{6}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \le i \le \delta), \tag{7}$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

We say the above TD pair is over $\mathbb{K}$.

We now show any Leonard pair is a Tridiagonal pair.

Lemma 2.2 *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ and $A^* : V \to V$ denote linear transformations. Then the following are equivalent.*

(i) A, A^* is a Leonard pair on V.

(ii) A, A^* is a TD pair on V, and for each of A, A^* all eigenspaces have dimension 1.

Proof: *(i)* $\to$ *(ii)* We show A and A^* satisfy the conditions (i)–(iv) of Definition 2.1. Condition (i) holds, since it is immediate from Definition 1.1 that A and A^* are diagonalizable on V. We now show A and A^* satisfy Definition 2.1(ii). Let $v_0, v_1, \ldots, v_d$ denote the basis for V referred to in Definition 1.1(i). For $0 \leq i \leq d$, let V_i denote the subspace of V spanned by v_i. By [54, Lem. 1.3] the eigenvalues of A associated with $v_0, v_1, \ldots, v_d$ are mutually distinct, so $V_0, V_1, \ldots, V_d$ is an ordering of the eigenspaces of A. This ordering satisfies (6), since the matrix representing A^* with respect to $v_0, v_1, \ldots, v_d$ is tridiagonal. We have now shown A, A^* satisfy Definition 2.1(ii). Interchanging the roles of A and A^*, we find A, A^* satisfy Definition 2.1(iii). It is immediate from [54, Lem. 3.3] that A, A^* satisfy Definition 2.1(iv). We have now shown A, A^* satisfy the conditions (i)–(iv) of Definition 2.1, so A, A^* is a TD pair on V. From our above comments the eigenspaces of A all have dimension 1. Similarly the eigenspaces of A^* all have dimension 1.
(ii) $\to$ *(i)* We show A and A^* satisfy conditions (i), (ii) of Definition 1.1. Concerning condition (i), let $V_0, V_1, \ldots, V_d$ denote the ordering of the eigenspaces of A referred to in Definition 2.1(ii). By assumption each of these eigenspaces has dimension 1. For $0 \leq i \leq d$, let v_i denote a nonzero element in V_i, and observe $v_0, v_1, \ldots, v_d$ is a basis for V. From the construction the matrix representing A with respect to this basis is diagonal. Using (6), we find the matrix representing A^* with respect to this basis is tridiagonal. This tridiagonal matrix is irreducible in view of Definition 2.1(iv). We have now shown A and A^* satisfy Definition 1.1(i). Interchanging the roles of A and A^*, we find A and A^* satisfy Definition 1.1(ii). It follows A, A^* is a Leonard pair on V.
□

Given the connection between Leonard pairs and the polynomials of the Askey scheme, we find it worthwhile to investigate TD pairs. We want to know if TD pairs correspond to multi-variable generalizations of the above-mentioned polynomials, and if not, what they do correspond to. TD pairs have a lot of structure and we believe a complete classification is possible. At present we have nothing of the sort so we pose the following problem.

Problem 2.3 *Classify or describe the Tridiagonal pairs.*

In [33] we obtained the following results on TD pairs. Referring to the TD pair in Definition 2.1, we showed $d = \delta$; we call this common value the *diameter* of the pair. We showed that for $0 \leq i \leq d$, the eigenspaces V_i and V_i^* have the same dimension. Denoting this common dimension by ρ_i, we showed the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is symmetric and unimodal; that is, $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. We also obtained some results of a representation theoretic nature, which we will discuss in the next section.

3 Tridiagonal pairs in representation theory

In this section we survey how TD pairs arise from irreducible finite dimensional modules of the Lie algebra sl_2, the Onsager algebra, the quantum algebra $U_q(sl_2)$, and the q-Onsager algebra. We define an algebra on two generators which we call the *Tridiagonal algebra*. This algebra generalizes both the Onsager and q-Onsager algebra. We show every TD pair comes from an irreducible finite dimensional module of a Tridiagonal algebra. For a more detailed discussion of the results in this section see [33].

We begin with sl_2.

Example 3.1 *[33, Ex. 1.5] Assume $\mathbb{K}$ is algebraically closed with characteristic 0, and let L denote the Lie algebra $sl_2(\mathbb{K})$. Let A and A^* denote semi-simple elements in L and assume L is generated by these elements. Let V denote an irreducible finite dimensional L-module. Then A and A^* act on V as a Leonard pair.*

Referring to the previous example, it is not hard to show A and A^* must satisfy

$$[A,[A,[A,A^*]]] = b^2[A,A^*], \tag{8}$$

$$[A^*,[A^*,[A^*,A]]] = b^{*2}[A^*,A], \tag{9}$$

where [,] denotes the Lie bracket, and where b and b^* denote nonzero scalars in $\mathbb{K}$. The relations (8), (9) are known in statistical mechanics as the *Dolan-Grady relations* [15], [16], [17], [45], [55].

Using the Dolan-Grady relations, we obtain TD pairs which are not necessarily Leonard pairs.

Example 3.2 *[33] Assume $\mathbb{K}$ is algebraically closed with characteristic 0. Let b and b^* denote nonzero scalars in $\mathbb{K}$. Let O denote the Lie algebra over $\mathbb{K}$ generated by symbols A, A^* subject to the Dolan-Grady relations (8), (9). Let V denote an irreducible finite dimensional O-module. Then A, A^* act on V as a TD pair.*

Referring to the above example, the algebra O is known in statistical mechanics as the Onsager algebra [1], [14], [15], [16], [44], [45], [55]. See Davies [16], Roan [45], and Date and Roan [15] for a detailed description of the irreducible finite dimensional O-modules.

Our next two examples are q-analogs of the previous two. To set the stage, we recall the quantum algebra $U_q(sl_2)$ and its modules. For notational convenience, we will replace q by $q^{1/2}$ and consider $U_{q^{1/2}}(sl_2)$.

Definition 3.3 *[34, p.122] Assume $\mathbb{K}$ is algebraically closed, and let q denote a nonzero scalar in $\mathbb{K}$ which is not a root of unity. Let $U_{q^{1/2}}(sl_2)$ denote the associative $\mathbb{K}$-algebra with 1 generated by symbols e, f, k, k^{-1} subject to the relations*

$$kk^{-1} = k^{-1}k = 1,$$

$$ke = qek, \qquad kf = q^{-1}fk,$$

$$ef - fe = \frac{k-k^{-1}}{q^{1/2}-q^{-1/2}}.$$

We now recall the irreducible finite dimensional modules for $U_{q^{1/2}}(sl_2)$.

Lemma 3.4 *[34, p. 128] With reference to Definition 3.3, there exists a family*

$$V_{\varepsilon,d} \qquad \varepsilon \in \{1,-1\}, \qquad d = 0,1,2\ldots \tag{10}$$

of irreducible finite dimensional $U_{q^{1/2}}(sl_2)$ modules with the following properties. The module $V_{\varepsilon,d}$ has a basis $v_0, v_1, \ldots, v_d$ satisfying $kv_i = \varepsilon q^{d/2-i} v_i$ for $0 \le i \le d$, $fv_i = [i+1]_q v_{i+1}$ for $0 \le i \le d-1$, $fv_d = 0$, $ev_i = \varepsilon[d-i+1]_q v_{i-1}$ for $1 \le i \le d$, $ev_0 = 0$, where

$$[i]_q = \frac{q^{i/2} - q^{-i/2}}{q^{1/2} - q^{-1/2}} \qquad i \in \mathbb{Z}. \tag{11}$$

Every irreducible finite dimensional module for $U_{q^{1/2}}(sl_2)$ is isomorphic to exactly one of the modules (10). (Referring to line (10), if $\mathbb{K}$ has characteristic 2 we interpret the set $\{1,-1\}$ as having a single element).

The following result was proved by the present author in [53] and is implicit in the results of Koelink and Jan Der. Keugt [38], [39].

Example 3.5 *[38], [39], [53] Referring to Definition 3.3 and Lemma 3.4, let α, α^* denote nonzero scalars in $\mathbb{K}$, and set*

$$A = \alpha f + \frac{k}{q^{1/2} - q^{-1/2}},$$

$$A^* = \alpha^* e + \frac{k^{-1}}{q^{1/2} - q^{-1/2}}.$$

Let d denote a nonnegative integer and pick $\varepsilon \in \{1,-1\}$. Then A, A^* act on $V_{\varepsilon,d}$ as a Leonard pair provided $\varepsilon\alpha\alpha^*$ is not among $q^{(d-1)/2}, q^{(d-3)/2}, \ldots, q^{(1-d)/2}$.

For related results concerning $U_{q^{1/2}}(sl_2)$ see [36], [37], [40], [46, ch. 4].

Referring to Example 3.5, one can show A and A^* satisfy

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0, \tag{12}$$
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0, \tag{13}$$

where $[3]_q = q + q^{-1} + 1$. The equations (12), (13) are known as the *q-Serre relations*, and are among the defining relations for the quantum affine algebra $U_{q^{1/2}}(\widehat{sl}_2)$. See [7], [8] for more information about this algebra.

Using the q-Serre relations, we obtain TD pairs that are not necessarily Leonard pairs.

Example 3.6 *[33, Ex. 1.7] Assume $\mathbb{K}$ is algebraically closed, and let q denote a nonzero element of $\mathbb{K}$ which is not a root of unity. Let O_q denote the associative $\mathbb{K}$-algebra with 1 generated by symbols A, A^* subject to the q-Serre relations (12), (13). Let V denote an irreducible finite dimensional module for O_q, and assume neither A nor A^* acts nilpotently on V. Then A, A^* act on V as a TD pair.*

Referring to the above example, we call O_q the *q-Onsager algebra*.

The TD pairs that come from the above four examples do not exhaust all TD pairs. To get all TD pairs, we consider a pair of relations which generalize both the Dolan-Grady relations and the q-Serre relations. We call these the *Tridiagonal relations*. In [33] we proved the following result for an arbitrary field $\mathbb{K}$.

Theorem 3.7 *[33] Let A, A^* denote a TD pair over $\mathbb{K}$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathbb{K}$ such that*

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] = 0, \qquad (14)$$
$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A] = 0, \qquad (15)$$

where $[r, s]$ means $rs - sr$. The sequence $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ is uniquely determined by the pair if the diameter is at least 3. We refer to $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ as a parameter sequence for A, A^*.

We call (14), (15) the *Tridiagonal relations* (or *TD relations*).

Remark 3.8 *The Dolan-Grady relations (8), (9) are the TD relations with parameters $\beta = 2$, $\gamma = \gamma^* = 0$, $\varrho = b^2$, $\varrho^* = b^{*2}$, if we interpret the bracket in (8), (9) as $[r, s] = rs - sr$. The q-Serre relations (12), (13) are the TD relations with parameters $\beta = q + q^{-1}$, $\gamma = \gamma^* = 0$, $\varrho = \varrho^* = 0$.*

Definition 3.9 *[33], [54] Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote a sequence of scalars taken from $\mathbb{K}$. We let $T = T(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$ denote the associative $\mathbb{K}$-algebra with 1 generated by two symbols A, A^* subject to the Tridiagonal relations (14), (15). We refer to T as the Tridiagonal algebra (or TD algebra) over $\mathbb{K}$ with parameters $\beta, \gamma, \gamma^*, \varrho, \varrho^*$. We refer to A and A^* as the standard generators of T.*

Reformulating Theorem 3.7 using the above definition, we can say the following. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^* denote a TD pair on V with parameter sequence $\beta, \gamma, \gamma^*, \varrho, \varrho^*$. By Theorem 3.7 the maps A, A^* induce on V a module structure for the algebra $T(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$. This module is irreducible in view of Definition 2.1(iv).

Given our above comments, the reader might wonder if every irreducible finite dimensional module for a TD algebra gives a TD pair. It turns out this is not quite true. Our result is the following.

Theorem 3.10 *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote scalars in $\mathbb{K}$, and assume q is not a root of unity, where $q + q^{-1} = \beta$. Let T denote the TD algebra over $\mathbb{K}$ with parameters $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ and standard generators A, A^*. Let V denote an irreducible finite dimensional T-module and assume each of A, A^* is diagonalizable on V. Then A, A^* act on V as a TD pair.*

Proof: We show A and A^* satisfy conditions (i)–(iv) of Definition 2.1. Condition (i) holds by assumption, so consider condition (ii). Let Ω denote the set of distinct eigenvalues of A on V. For $\theta \in \Omega$, let V_θ denote the corresponding eigenspace for A and let E_θ denote the

projection of V onto V_θ. We recall $E_\theta - I$ vanishes on V_θ, where I denotes the identity map on V, and that E_θ vanishes on V_μ for all $\mu \in \Omega$, $\mu \neq \theta$. Let θ, μ denote distinct elements of Ω. We determine when $E_\theta A^* E_\mu$ is zero. To do this, we introduce the following polynomial in two variables x, y.

$$p(x,y) = x^2 - \beta xy + y^2 - \gamma(x+y) - \varrho. \tag{16}$$

We now assume $E_\theta A^* E_\mu \neq 0$ and show $p(\theta, \mu) = 0$. For notational convenience, set

$$C = A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*,$$

and observe $AC = CA$ by (14). Using $E_\theta A = \theta E_\theta$ and $AE_\mu = \mu E_\mu$, we have

$$\begin{aligned} 0 &= E_\theta(AC - CA)E_\mu \\ &= (\theta - \mu)E_\theta C E_\mu, \end{aligned}$$

and since $\theta \neq \mu$,

$$\begin{aligned} 0 &= E_\theta C E_\mu \\ &= E_\theta A^* E_\mu(\theta^2 - \beta\theta\mu + \mu^2 - \gamma(\theta + \mu) - \varrho) \\ &= E_\theta A^* E_\mu p(\theta, \mu). \end{aligned}$$

We assumed $E_\theta A^* E_\mu \neq 0$, so $p(\theta, \mu) = 0$ as desired. For all $\theta, \mu \in \Omega$, we define θ and μ to be *adjacent* whenever $\theta \neq \mu$ and $p(\theta, \mu) = 0$. Since the polynomial p is quadratic in its arguments, each element in Ω is adjacent at most two elements of Ω. We show the adjacency relation has no "cycles". Let r denote an integer at least 3. We claim there does not exist a sequence $\theta_1, \theta_2, \ldots, \theta_r$ consisting of distinct elements of Ω, such that θ_i is adjacent θ_{i+1} for $1 \leq i < r$ and θ_r is adjacent θ_1. Suppose such a sequence exists and consider the infinite sequence

$$\theta_1, \theta_2, \ldots, \theta_r, \theta_1, \theta_2, \ldots, \theta_r, \ldots \tag{17}$$

For $i = 0, 1, 2, \ldots$ let σ_i denote the i^{th} term in this sequence. For $i = 1, 2, \ldots$ observe σ_{i-1} and σ_{i+1} are distinct and adjacent σ_i, so they are the roots of $p(x, \sigma_i)$. Apparently $\sigma_{i-1} + \sigma_{i+1}$ is the opposite of the coefficient of x in $p(x, \sigma_i)$. Setting $y = \sigma_i$ in (16), we see this coefficient is $-\beta\sigma_i - \gamma$, so

$$\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma \qquad i = 1, 2, \ldots \tag{18}$$

By definition $\beta = q + q^{-1}$, and we assume $q \notin \{1, -1\}$, so $\beta \notin \{2, -2\}$. Solving the linear recurrence (18) using this, we find there exists scalars a, b, c in the algebraic closure of $\mathbb{K}$ such that

$$\sigma_i = a + bq^i + cq^{-i} \qquad i = 0, 1, 2, \ldots \tag{19}$$

We show (19) is inconsistent with the periodic nature of (17). For $i = 0, 1, 2, \ldots$ we have $\sigma_i = \sigma_{i+r}$. Using (19) we find $\sigma_i - \sigma_{i+r}$ equals $1 - q^r$ times

$$bq^i - cq^{-i-r}. \tag{20}$$

We assume $q^r \neq 1$, so the expression (20) is zero. Setting the expression (20) equal to zero for $i = 0, 1$ and using $q \neq 1$, $q \neq -1$, we routinely find $b = 0, c = 0$. But this is inconsistent

with (19) and the requirement $\sigma_0 \neq \sigma_1$. We have now proved the claim. From our above remarks, there exists an ordering $\theta_0, \theta_1, \ldots, \theta_d$ of all the elements of Ω such that θ_i, θ_j are not adjacent for $|i-j| > 1$, $(0 \leq i, j \leq d)$. From our preliminary comments and abbreviating E_i for E_{θ_i},

$$E_i A^* E_j = 0 \quad \text{if} \quad |i-j| > 1, \qquad\qquad (0 \leq i, j \leq d).$$

From the above line and abbreviating V_i for V_{θ_i}, we obtain

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad\qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$. We now see A, A^* satisfy Definition 2.1(ii). Interchanging the roles of A and A^* in the above argument, we see A, A^* satisfy Definition 2.1(iii). Observe A, A^* satisfy Definition 2.1(iv), since we assume V is irreducible as a T-module. We have now shown A, A^* satisfy conditions (i)–(iv) of Definition 2.1, so A, A^* act on V as a TD pair. $\square$

We finish this section with a few comments on the Tridiagonal relations (14), (15). These relations previously appeared in [51]. In that paper, the author considers a combinatorial object called a P- and Q-polynomial association scheme [4], [5], [42], [48], [52]. He shows that for these schemes the adjacency matrix A and a certain diagonal matrix A^* satisfy (14), (15). In this context the algebra generated by A and A^* is known as the subconstituent algebra or the Terwilliger algebra [49], [50], [51]. See [6], [9], [10], [11], [12], [13], [19], [32], [47] for more information on this algbra.

We mention the relations (14), (15) are satisfied by the generators of both the classical and quantum Quadratic Askey-Wilson algebra introduced by Granovskii, Lutzenko, and Zhedanov [21]. See [20], [22], [23], [24], [56], [57], [58] for more information on this algebra.

The relation (15) previously appeared in the work of Grunbaum and Haine on the "bispectral problem" [27], [28]. See [25], [26], [29], [30], [31] for related work.

4 The parameter sequence and the eigenvalues

In this section we obtain the following results concerning the parameter sequence of a TD pair. We consider a transformation on TD pairs and determine the effect on the parameter sequence. We recall the eigenvalue and dual eigenvalue sequences of a TD pair, and show these sequences satisfy a three term recurrence. We show how to get the parameter sequence of a TD pair from its eigenvalue and dual eigenvalue sequences. We characterize the TD pairs satisfying the Dolan-Grady relations in terms of their eigenvalue sequence and dual eigenvalue sequence. We obtain a similar characterization of the TD pairs satisfying the q-Serre relations.

We begin with an observation. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension, and let A, A^* denote a TD pair on V. Let r, s, r^*, s^* denote scalars in $\mathbb{K}$ with each of r, r^* nonzero. Then the ordered pair

$$rA + sI, \qquad r^*A^* + s^*I \tag{21}$$

is a TD pair on V. The above transformation has the following effect on the parameter sequence. Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote a parameter sequence for A, A^*, as in Theorem 3.7. Then the following is a parameter sequence for the TD pair (21).

$$\begin{array}{lll} \beta, & r\gamma + s(2-\beta), & r^*\gamma^* + s^*(2-\beta), \\ r^2\varrho - 2rs\gamma + s^2(\beta-2), & & r^{*2}\varrho^* - 2r^*s^*\gamma^* + s^{*2}(\beta-2). \end{array}$$

A given parameter sequence can often be simplified by transforming it in the above fashion. How far one can go in this direction depends on the field $\mathbb{K}$, so we will not give all the details. Instead, we illustrate with an example. Let us say a parameter sequence $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ is *reduced* whenever $\gamma = \gamma^* = 0$. Let A, A^* denote a TD pair on V with parameter sequence $\beta, \gamma, \gamma^*, \varrho, \varrho^*$. Then the TD pair below has a reduced parameter sequence provided $\beta \neq 2$.

$$A + \gamma(\beta-2)^{-1}I, \qquad\qquad A^* + \gamma^*(\beta-2)^{-1}I.$$

We now consider the relationship between the parameter sequence and the eigenvalues of A and A^*.

Definition 4.1 *[33] Let A, A^* denote a TD pair with diameter d. By an eigenvalue sequence for A, A^*, we mean an ordering $\theta_0, \theta_1, \ldots, \theta_d$ of the distinct eigenvalues of A that satisfies Definition 2.1(ii), where for $0 \leq i \leq d$ the V_i in that definition denotes the eigenspace of A associated with θ_i. We remark that if $\theta_0, \theta_1, \ldots, \theta_d$ is an eigenvalue sequence for A, A^* then so is $\theta_d, \theta_{d-1}, \ldots, \theta_0$, and A, A^* has no further eigenvalue sequence. By a dual eigenvalue sequence for A, A^*, we mean an eigenvalue sequence for A^*, A.*

In [33] we obtained the following result.

Theorem 4.2 *[33] let A, A^* denote a TD pair, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$. Then the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad\qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{22}$$

are equal and independent of i for $2 \leq i \leq d-1$.

In the following theorem, we explain how to obtain a parameter sequence for a given TD pair from its eigenvalue sequence and dual eigenvalue sequence. To prepare for this result we make some comments. For the moment, let $\theta_0, \theta_1, \ldots, \theta_d$ denote any finite sequence of distinct scalars in $\mathbb{K}$, and assume

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \tag{23}$$

is independent of i for $2 \leq i \leq d-1$. Denoting by $\beta + 1$ the common value of the expressions (23), we find $\theta_{i-1} - \beta\theta_i + \theta_{i+1}$ is independent of i for $1 \leq i \leq d-1$. Denoting this common value by γ, we have

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad\qquad (1 \leq i \leq d-1).$$

We claim there exists a scalar $\varrho \in \mathbb{K}$ such that

$$\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \qquad (1 \leq i \leq d). \tag{24}$$

To see this, let p_i denote the expression on the right in (24), and observe

$$p_i - p_{i+1} = (\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta\theta_i + \theta_{i+1} - \gamma) \tag{25}$$

for $1 \leq i \leq d-1$. From (25) we see p_i is independent of i for $1 \leq i \leq d$. Denoting the common value of the p_i by ϱ, we obtain (24). With these comments in mind, we present the next result.

Theorem 4.3 *[33] Let A, A^* denote a TD pair over $\mathbb{K}$, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote a sequence of scalars taken from $\mathbb{K}$. Then this is a parameter sequence for A, A^* if and only if (i)–(v) hold below.*

(i) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

both equal $\beta + 1$ for $2 \leq i \leq d-1$.

(ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad (1 \leq i \leq d-1)$,

(iii) $\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \qquad (1 \leq i \leq d-1)$,

(iv) $\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \qquad (1 \leq i \leq d)$,

(v) $\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \qquad (1 \leq i \leq d)$.

To get another point of view, we consider the eigenvalues and dual eigenvalues in parametric form. Solving the linear recurrence in Theorem 4.2, we obtain the following.

Theorem 4.4 *[33] Let A, A^* denote a TD pair over $\mathbb{K}$, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Then these sequences are given by Case I, II, or III below.*

Case I:

$$\begin{aligned} \theta_i &= a + bq^i + cq^{-i}, & \qquad q \neq 0, \quad q \neq 1, \quad q \neq -1 \\ \theta_i^* &= a^* + b^*q^i + c^*q^{-i}. & \end{aligned}$$

Case II:

$$\begin{aligned} \theta_i &= a + bi + ci(i-1)/2, \\ \theta_i^* &= a^* + b^*i + c^*i(i-1)/2. \end{aligned}$$

Case III: The characteristic of $\mathbb{K}$ is not 2, and

$$\begin{aligned} \theta_i &= a + b(-1)^i + ci(-1)^i, \\ \theta_i^* &= a^* + b^*(-1)^i + c^*i(-1)^i. \end{aligned}$$

In the above formulae, the scalars q and a, b, c, a^*, b^*, c^* are in the algebraic closure of $\mathbb{K}$. Concerning Case II, if the characteristic of $\mathbb{K}$ equals 2, we interpret the expression $i(i-1)/2$ as 0 if $i = 0$ or $i = 1$ (mod 4), and as 1 if $i = 2$ or $i = 3$ (mod 4).

Evaluating the data in Theorem 4.3 using Theorem 4.4, we obtain the following.

Lemma 4.5 *Referring to Theorem 4.4, the sequence $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ given below is a parameter sequence for A, A^*.*

Case I:

$$\begin{aligned} \beta &= q + q^{-1}, \\ \gamma &= -a(q-1)^2 q^{-1}, \\ \varrho &= a^2(q-1)^2 q^{-1} - bc(q-q^{-1})^2. \end{aligned}$$

Case II:

$$\begin{aligned} \beta &= 2, \\ \gamma &= c, \\ \varrho &= b^2 - bc - 2ac. \end{aligned}$$

Case III:

$$\begin{aligned} \beta &= -2, \\ \gamma &= 4a, \\ \varrho &= c^2 - 4a^2. \end{aligned}$$

To get γ^*, ϱ^*, replace a, b, c in the above lines by a^*, b^*, c^*.

At the beginning of this section we defined the notion of a reduced parameter sequence. Concerning this we have the following result.

Corollary 4.6 *The parameter sequence given in Lemma 4.5 is reduced if and only if the following hold.*

Cases I, III: $\quad a = 0, \ \ a^* = 0.$

Case II: $\quad c = 0, \ \ c^* = 0.$

We finish this section with some comments on the Dolan-Grady relations and the q-Serre relations. For the moment let $\theta_0, \theta_1, \ldots, \theta_d$ denote any finite sequence of scalars in $\mathbb{K}$. Let b denote a scalar in $\mathbb{K}$. We say the sequence $\theta_0, \theta_1, \ldots, \theta_d$ is in *b-arithmetic progression* whenever $\theta_i = \theta_{i-1} + b$ for $1 \leq i \leq d$. Let q denote a nonzero scalar in $\mathbb{K}$. We say $\theta_0, \theta_1, \ldots, \theta_d$ is in *q-geometric progression* whenever $\theta_i = \theta_{i-1} q$ for $1 \leq i \leq d$.

Lemma 4.7 *Let A, A^* denote a TD pair over $\mathbb{K}$ and let b, b^* denote nonzero scalars in $\mathbb{K}$. Then (i), (ii) below are equivalent.*

(i) A, A^* satisfy the Dolan-Grady relations (8), (9) for b and b^*.

(ii) There exists an eigenvalue sequence for A, A^* which is in b-arithmetic progression, and there exists a dual eigenvalue sequence for A, A^* which is in b^*-arithmetic progression.

Proof: $(i) \to (ii)$ By Remark 3.8 we see $2, 0, 0, b^2, b^{*2}$ is a parameter sequence for A, A^*. Applying Theorem 4.3 to this sequence, we routinely obtain the result.
$(ii) \to (i)$. Setting $c = 0$, $c^* = 0$ in Case II of Lemma 4.5, we find $2, 0, 0, b^2, b^{*2}$ is a parameter sequence for A, A^*. The result now follows in view of Remark 3.8.
□

Lemma 4.8 *Let A, A^* denote a TD pair over $\mathbb{K}$ and let q denote a nonzero scalar in $\mathbb{K}$ other than $1, -1$. Then (i), (ii) below are equivalent.*

(i) A, A^* satisfy the q-Serre relations (12), (13).

(ii) There exists an eigenvalue sequence for A, A^* which is in q-geometric progression, and there exists a dual eigenvalue sequence for A, A^* which is in q-geometric progression.

Proof: Similar to the proof of Lemma 4.7.
□

5 Infinite dimensional modules for Tridiagonal algebras

In this section we consider infinite dimensional irreducible modules for TD algebras. These modules seem more complicated than the finite dimensional ones, so we do not attempt a general theory. Instead, we motivate further study by presenting two examples obtained from certain orthogonal polynomials of the Askey scheme. Our first example, based on the Hermite polynomials, is very simple and meant to illustrate the ideas involved. Our second and main example is based on the Askey-Wilson polynomials. We would like to acknowledge that the results of this section can be readily obtained from any of the following works: Granovskii, Lutzenko, and Zhedanov [21], Grunbaum and Haine [27], Noumi and Stokman [43]. Our goal here is to emphasize the role played by the Tridiagonal relations.

Throughout this section x will denote an indeterminant and $\mathbb{K}[x]$ will denote the $\mathbb{K}$-algebra consisting of all polynomials in x that have coefficients in $\mathbb{K}$.

Example 5.1. **The Hermite polynomials**. For this example assume $\mathbb{K}$ has characteristic 0. Let $H_0, H_1, \ldots$ denote the polynomials in $\mathbb{K}[x]$ satisfying

$$xH_n = H_{n+1} + 2nH_{n-1}, \qquad n = 0, 1, 2, \ldots \tag{26}$$

where $H_0 = 1$, $H_{-1} = 0$. We refer to H_n as the n^{th} *Hermite polynomial* [35]. The first four Hermite polynomials are

$$H_0 = 1, \qquad H_1 = x, \qquad H_2 = x^2 - 2, \qquad H_3 = x^3 - 6x.$$

The Hermite polynomials satisfy a second order differential equation, obtained as follows. Let $D : \mathbb{K}[x] \to \mathbb{K}[x]$ denote the derivative map. We recall D is a linear transformation satisfying $Dx^n = nx^{n-1}$ for $n = 0, 1, 2, \ldots$ It is well known

$$DH_n = nH_{n-1} \qquad n = 0, 1, 2 \ldots \tag{27}$$

This can be proved by induction on n, or see [2, p. 280]. Combining (26), (27), we find

$$(x - 2D)H_n = H_{n+1} \qquad\qquad n = 0, 1, 2 \ldots \tag{28}$$

Combining (27), (28), we obtain

$$(x - 2D)DH_n = nH_n \qquad\qquad n = 0, 1, 2 \ldots \tag{29}$$

This is the differential equation satisfied by the Hermite polynomials. We now endow the vector space $\mathbb{K}[x]$ with a module structure for a certain TD algebra. Let A and A^* denote the following linear transformations.

$$\begin{array}{cccc} & \mathbb{K}[x] & \rightarrow & \mathbb{K}[x] \\ A: & f & \rightarrow & xf, \\ A^*: & f & \rightarrow & (x-2D)Df. \end{array}$$

With respect to the basis $H_0, H_1, \ldots$ the matrices representing A and A^* are, respectively,

$$\left(\begin{array}{ccccc} 0 & 2 & & & \\ 1 & 0 & 4 & & \\ & 1 & 0 & 6 & \\ & & 1 & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & \end{array}\right), \qquad \mathrm{diag}(0, 1, 2, \ldots). \tag{30}$$

These are obtained from (26) and (29). Representing A and A^* by the matrices in (30), we routinely find

$$A^2A^* - 2AA^*A + A^*A^2 = -4I, \tag{31}$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - A = 0, \tag{32}$$

where I denotes the identity map on $\mathbb{K}[x]$. Observe the left side of (31) commutes with A and the left side of (32) commutes with A^*. Therefore A and A^* satisfy the TD relations (14), (15) for $\beta = 2$, $\gamma = \gamma^* = 0$, $\delta = 0$, $\delta^* = 1$. Apparently A and A^* induce a module structure on $\mathbb{K}[x]$ for the algebra $T(2, 0, 0, 0, 1)$. This module is irreducible in view of the following lemma. The proof of this lemma is routine and omitted.

Lemma 5.2 *Let V denote an infinite dimensional vector space over $\mathbb{K}$. Let $A : V \rightarrow V$ and $A^* : V \rightarrow V$ denote linear transformations. Assume V has a basis $v_0, v_1, \ldots$ with respect to which (i) the matrix representing A is irreducible tridiagonal, and (ii) the matrix representing A^* is diagonal, with diagonal entries mutually distinct. Then there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$, other than $W = 0$ and $W = V$.*

Example 5.3. **The Askey-Wilson polynomials**. For this example assume the field $\mathbb{K}$ is arbitrary. Let q, a, b, c, d denote nonzero scalars in $\mathbb{K}$. To avoid degenerate situations, we assume q is not a root of unity, and that none of $ab, ac, ad, bc, bd, cd, abcd$ is an integral power of q. For $n = 0, 1, 2 \ldots$ let $p_n = p_n(x; a, b, c, d)$ denote the polynomial in $\mathbb{K}[x]$ given by

$$p_n = {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{array} \,\middle|\, q; q\right),$$

where $x = y + y^{-1}$. We follow the standard notation for basic hypergeometric series given in [18]. We refer to p_n as the n^{th} *Askey-Wilson polynomial* with parameters a, b, c, d [3], [18], [35]. The first three Askey-Wilson polynomials are $p_0 = 1$,

$$\begin{aligned} p_1 &= 1 - \frac{(1-abcd)(1-ax+a^2)}{(1-ab)(1-ac)(1-ad)}, \\ p_2 &= 1 - \frac{(1+q^{-1})(1-abcdq)(1-ax+a^2)}{(1-ab)(1-ac)(1-ad)} \\ &\qquad + \frac{(1-abcdq)(1-abcdq^2)(1-ax+a^2)(1-axq+a^2q^2)}{q(1-ab)(1-abq)(1-ac)(1-acq)(1-ad)(1-adq)}. \end{aligned}$$

The Askey-Wilson polynomials satisfy the following three term recurrence [3]. For $n = 0, 1, 2, \ldots$ we have

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \tag{33}$$

where $p_{-1} = 0$, and where

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \tag{34}$$

$$c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}, \tag{35}$$

$$a_n = a + a^{-1} - b_n - c_n. \tag{36}$$

We observe the denominators in (34), (35) are not zero, and that $b_{n-1}c_n \neq 0$ for $n = 1, 2, \ldots$ The Askey-Wilson polynomials satisfy a q-difference equation, obtained as follows. Let y denote an indeterminant, and let $\mathbb{K}[y, y^{-1}]$ denote the $\mathbb{K}$-algebra consisting of all Laurent polynomials in y that have coefficients in $\mathbb{K}$. Identifying x with $y + y^{-1}$, we view $\mathbb{K}[x]$ as the subalgebra of $\mathbb{K}[y, y^{-1}]$ generated by x. From this point of view $\mathbb{K}[x]$ has basis

$$1, \quad y + y^{-1}, \quad y^2 + y^{-2}, \quad y^3 + y^{-3}, \quad \ldots$$

Let τ denote the $\mathbb{K}$-algebra automorphism of $\mathbb{K}[y, y^{-1}]$ satisfying $\tau(y) = qy$. Let $\mathbb{D}$ denote the restriction of the following map to $\mathbb{K}[x]$:

$$\phi(y)(\tau - I) \; + \; \phi(y^{-1})(\tau^{-1} - I) \; + \; (1 + abcdq^{-1})I. \tag{37}$$

Here I denotes the identity map and

$$\phi(y) = \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)}.$$

In concrete terms, $\mathbb{D} : \mathbb{K}[x] \to \mathbb{K}[x]$ is the linear transformation satisfying $\mathbb{D}(1) = 1 + abcdq^{-1}$, and for $n = 1, 2, 3 \ldots$

$$\begin{aligned} \mathbb{D}(y^n + y^{-n}) &= (1 + abcdq^{-1})(y^n + y^{-n}) \\ &\quad + \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)}(q^n y^n - y^n + q^{-n}y^{-n} - y^{-n}) \\ &\quad + \frac{(1-ay^{-1})(1-by^{-1})(1-cy^{-1})(1-dy^{-1})}{(1-y^{-2})(1-qy^{-2})}(q^{-n}y^n - y^n + q^n y^{-n} - y^{-n}). \end{aligned}$$

The map $\mathbb{D}$ is known as the *Askey-Wilson second order q-difference operator* with parameters a, b, c, d [3]. The Askey-Wilson polynomials mentioned above are the eigenvectors for $\mathbb{D}$. In fact, for $n = 0, 1, 2, \ldots$ we have

$$\mathbb{D}\, p_n = \theta^*_n\, p_n, \tag{38}$$

where

$$\theta^*_n = q^{-n} + abcdq^{n-1}. \tag{39}$$

For a proof of this see [3]. Line (38) is the above-mentioned q-difference equation satisfied by the Askey-Wilson polynomials. We now endow the vector space $\mathbb{K}[x]$ with the module structure of a certain TD algebra. Let A and A^* denote the following linear transformations.

$$\begin{array}{rccc} & \mathbb{K}[x] & \to & \mathbb{K}[x] \\ A: & f & \to & xf, \\ A^*: & f & \to & \mathbb{D}f. \end{array}$$

We show A and A^* satisfy the TD relations

$$\begin{aligned} [A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + (q-q^{-1})^2A^*] &= 0, & (40)\\ [A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} + abcdq^{-1}(q-q^{-1})^2A] &= 0, & (41) \end{aligned}$$

where we recall $[r, s]$ means $rs - sr$. We begin with (41). Observe $p_0, p_1, \ldots$ is a basis for the vector space $\mathbb{K}[x]$. With respect to this basis, the matrices representing A and A^* are

$$\left(\begin{array}{ccccc} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{array}\right), \qquad \mathrm{diag}(\theta^*_0, \theta^*_1, \theta^*_2, \ldots), \tag{42}$$

respectively, where the a_n, b_n, c_n are from (34)–(36), and where the θ^*_n are from (39). Let Δ denote the matrix representing the left side of (41) with respect to the basis $p_0, p_1, \ldots$ We show $\Delta = 0$. To do this, we show each entry of Δ is zero. Using the matrices (42) we find that for nonnegative integers r, s, the matrix Δ has r, s entry

$$\theta^{*2}_r - (q+q^{-1})\theta^*_r\theta^*_s + \theta^{*2}_s + abcdq^{-1}(q-q^{-1})^2 \tag{43}$$

times $(\theta^*_r - \theta^*_s)B_{rs}$, where B denotes the matrix on the left in (42). Observe B is tridiagonal, so $B_{rs} = 0$ for $|r-s| > 1$. In view of (39) the expression (43) is zero for $|r-s| = 1$. Of course $\theta^*_r - \theta^*_s$ is zero for $r = s$. Apparently Δ has all entries zero, so $\Delta = 0$, and (41) follows. We now show (40). It is possible to verify (40) using the above matrix representations of A and A^*, but the calculation is tedious. Instead, we follow the argument of Grunbaum and Haine [27] who obtained (40) in a different context. To obtain (40), we show A commutes with

$$A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + (q-q^{-1})^2A^*. \tag{44}$$

To do this, we recall the map A represents multiplication by x, and A^* represents the Askey-Wilson operator $\mathbb{D}$ given in (37). Consider the terms in (37). Recall the map τ in that line is the automorphism of $\mathbb{K}[y, y^{-1}]$ satisfying $\tau(y) = qy$. We show the map

$$x^2\tau - (q + q^{-1})x\tau x + \tau x^2 + (q - q^{-1})^2\tau \tag{45}$$

vanishes on $\mathbb{K}[y, y^{-1}]$. For $f \in \mathbb{K}[y, y^{-1}]$, the image of f under the map (45) equals

$$x^2 - (q + q^{-1})x\tau(x) + \tau(x)^2 + (q - q^{-1})^2 \tag{46}$$

times $\tau(f)$. Observe the expression (46) equals

$$(y + y^{-1})^2 - (q + q^{-1})(y + y^{-1})(qy + q^{-1}y^{-1}) + (qy + q^{-1}y^{-1})^2 + (q - q^{-1})^2$$

which equals zero, so the map (45) vanishes on $\mathbb{K}[y, y^{-1}]$. Similarly the map

$$x^2\tau^{-1} - (q + q^{-1})x\tau^{-1}x + \tau^{-1}x^2 + (q - q^{-1})^2\tau^{-1} \tag{47}$$

vanishes on $\mathbb{K}[y, y^{-1}]$. Replacing A^* by $\mathbb{D}$ in (44), and evaluating the result using (37) and our comments above, we find the map (44) is of the form ωI, where ω is an element of $\mathbb{K}[x]$. We now see A commutes with (44), as desired. We have now shown (40). By (40), (41), we find A and A^* induce on $\mathbb{K}[x]$ a module structure for the TD algebra with parameters $\beta = q + q^{-1}$, $\gamma = \gamma^* = 0$, $\varrho = -(q - q^{-1})^2$, $\varrho^* = -abcdq^{-1}(q - q^{-1})^2$. This module is irreducible in view of Lemma 5.2.

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Paul Terwilliger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, Wisconsin, 53706, USA
email: terwilli@math.wisc.edu